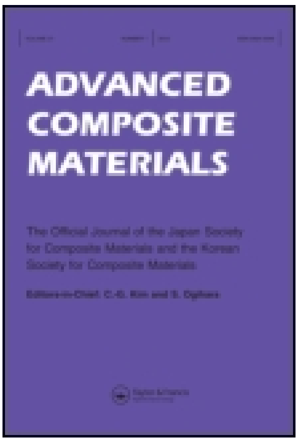

Advanced Composite Materials
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tacm20

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Version of record first published: 02 Apr 2012.

Adv. Composite Mater., Vol. 11, No. 4, pp. 331–350 (2003)

A C^0 element for free vibration of composite plates with uncertain material properties

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Received 26 March 2002; accepted 6 November 2002

Abstract—Randomness in the material properties is inherent in all engineering materials. Composites exhibit a greater scatter compared to metallic materials because of the larger number of parameters associated with their fabrication and manufacturing. For accurate prediction of their structural behavior, material properties have been modeled as random variables. The system equations have been derived based on higher order shear deformation theory and also include rotary inertia effects. The finite element method in conjunction with first order perturbation technique have been employed to obtain the second order statistics of free vibration response of composite laminated plates with random material properties. The results computed from the present approach are validated using the Monte-Carlo simulation. The standard deviation (SD) of the fundamental frequency is found to vary linearly with SD of the material properties. The influence of each random variable on the fundamental frequency has been examined.

Keywords: Composite plates; stochastic finite element method; random; free vibration; statistics.

1. INTRODUCTION

Fiber reinforced plastics (FRPs) consisting of strong and continuous fibers embedded in a polymer matrix have been developed over the last three decades. FRPs can be tailored to exhibit outstanding mechanical properties, such as high strength and stiffness-to-weight ratios, excellent corrosion resistance and very good fatigue characteristics. These have found use in a great variety of applications in aerospace, mechanical, civil, chemical, and other industries. Because of the ability to tailor their properties, these materials are being increasingly used in many new sensitive

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applications that open new challenges and invite new mathematical complexities for the analyst/designer.

The parameters characterizing the behavior of a composite laminate are many. Some of these may be enumerated as: lay up sequence, fiber orientation, individual lamina thickness, inter-laminar material, fiber volume fraction, voids, curing process, etc. Any variations in these parameters during manufacturing and fabrication result in variation in the behavior characteristics of the laminate.

The effectiveness of quality control during manufacturing and fabrication of a component depends on the strictness of the control and the number and types of parameters required to be controlled. Complete quality control is not practical for any material for economic reasons. The lack of complete control is reflected as random variation in the component characteristics, like geometrical and material properties. It can be seen that the variation in composite laminates is more pronounced than in components of conventional metallic materials, as they have more parameters involved in fabrication/manufacturing. Dispersions in composite material properties are further increased by variations in the material properties of the constituent components. On the whole, the material property variations for metallic materials are small, while composite laminates may have large variations.

As mentioned earlier, composite materials are used extensively in primary and secondary structures in aeronautical and space structures, like advanced aircrafts, helicopters, space stations, etc. Computer simulations of some of the proposed configurations of such aeronautical and space installations often show closely packed/overlapping natural frequencies in some of the components. In such cases, even the slightest shift in characteristics of the components can have a pronounced effect on the response of the structures.

The analysis of structures with deterministic characteristics to random excitations has been reported extensively in the literature. Nigam and Narayanan [1] have considered various types of loading in their studies. However, structural analysis with random material properties is not developed to the same extent.

Some literature is available on the analyses of conventional material structures with random material properties. Ibrahim [2, 3] has presented a review of structural dynamic problem with parameter uncertainties. Chen *et al.* [4] have developed a probabilistic method to evaluate the effect of uncertainty in geometrical and material properties for the truss and beam problems. The results for mean and standard deviation of displacement and rotation have been obtained. Collins and Thomson [5] have studied longitudinal vibration of four degrees of freedom fixed–fixed rod with uncertain area, mass and stiffness. A first order perturbation technique (FOPT) in stiffness has been employed to obtain the solutions. Shinozuka and Astill [6] have employed a numerical technique to obtain statistical properties of eigenvalues of spring-supported columns with deterministic axial loading and random material and geometrical properties, and the accuracy of FOPT has been examined. Caravani and Thomson [7] have studied the influence of damping uncertainty on frequency response of a multi-degree of freedom system. The linearized model has been adopted for second order descriptions. The limits of the validity of the approach have been investigated with Monte-Carlo simulation (MCS). Chen and Soroka [8] have studied the response of a multi-degrees of freedom system with random properties to deterministic excitations. FOPT has been employed for the study. Mean and SD of the response have been obtained. Liu *et al.* [9] have used probabilistic finite element technique to study the dynamic response of trusses with random geometry. The second order statistics of displacement and stress have been obtained and compared with MCS results. Zang and Chen [10] have presented a systematic approach for obtaining SD of eigenvalues and eigenvectors for a multi-degrees freedom random system using FOPT. Grigoriu [11] has developed a method for calculating the probabilistic characteristics of the eigenvalues of the stochastic symmetric matrices. The dynamics and elasticity problems have been considered to demonstrate the approach. Gorman [12] has investigated free vibration of thin rectangular plates in the domain of classical laminate theory with variable lateral edge support by the method of superposition.

Limited literature is available on the study of response of composite structures with random material properties. Leissa and Martin [13] have analyzed composite material panels with variable fiber spacing using classical laminate theory. Free vibration and buckling of flat plates have been analyzed, taking them to be macroscopically orthotropic but non-homogeneous because of variable fiber spacing. Results have been obtained for glass, boron and graphite fibers with epoxy matrix for simply supported square plates. Salim *et al.* [14–16] have employed first order perturbation technique (FOPT) to analyze composite plates using classical laminate theory with random material properties and have obtained second order statistics for static deflection, natural frequency and buckling loads in rectangular plates. The static response of graphite–epoxy composite laminates with randomness in material properties to deterministic loading have been obtained by Raj *et al.* [17] using combination of finite element method and MCS. Singh *et al.* [18, 19] have studied buckling and free vibration of composite laminates, with all edges simply supported, using FOPT. Higher order shear deformation theory (HSDT) has been employed to derive the governing system equation. Rotary inertia effects have also been incorporated in the analysis. The second order statistics of buckling load and natural frequencies have been computed. Further buckling analysis of composite plates with different boundary conditions using stochastic finite element method (SFEM) and HSDT has been attempted by Singh *et al.* [20]. Mean and standard deviation of buckling load have been evaluated.

From detailed examination of the literature, it is evident that free vibration study of shear deformable composite plates with different boundary conditions, including rotary inertia effect and having random material properties has received no attention. The aim of the present work is to obtain second order statistics of the fundamental frequency for composite plates with different boundary conditions. The lamina material properties have been modeled as random variables. The system equations in the framework of small deflection have been derived based on HSDT. Rotary inertia effects have been incorporated in the system equations. An approach is presented for solving free vibration of composite plates using the finite element method in conjunction with first order perturbation technique.

2. DETERMINISTIC FINITE ELEMENT FORMULATION

As stated above, there is need for an in-depth study to find the dispersion in free vibration response for a composite plate with various boundary conditions with random material properties for which a deterministic closed form solution is not available. It is also worthwhile from the design point of view to examine the effect of various geometric parameters on the response. Further, for proper quality control during manufacturing of the composite, it is required to investigate the sensitivity of the natural frequency to the variation in random material properties.

It is well known that, even in the case of thin composite laminates, the neglect of transverse shear strains makes the laminate stiffer. Since shear modulus is an independent material property in the composite, it would also affect the design. The classical laminate theory (CLT) developed by Smith [21], Reissner and Stavsky [22], and Lekhnitskii [23] were all based on Kirchoff's assumptions. The frequencies predicted using the CLT model were higher as compared to experimental observations. This was mainly due to the fact that the transverse shear stresses and rotatory inertia effects were neglected in the CLT model. Later, many studies were carried out to include these effects, which led to the development of various shear deformation theories [24–27]. The HSDT model developed by Reddy [28] for elastic plates has been employed for the present study. The present study is based on the so-called parabolic shear deformation theory.

2.1. Displacement field model

Consider a rectangular plate with Cartesian coordinates system. The origin of the rectangular, right-hand coordinates is located in the plate mid-plane at a corner with the x_1 and x_2 axes along the plate edges and ζ axis perpendicular to the mid-plane. The plate dimensions are taken as length a along x_1, width b along x_2 and thickness h along ζ directions. The displacement field relations are [28]:

$$\begin{aligned} \overline{u}(x_1, x_2, \zeta, t) &= u + \zeta\phi_1 + \zeta^2\varphi_1 + \zeta^3\theta_1, \\ \overline{v}(x_1, x_2, \zeta, t) &= v + \zeta\phi_2 + \zeta^2\varphi_2 + \zeta^3\theta_2, \quad \overline{w}(x_1, x_2, \zeta, t) = w, \end{aligned} \tag{1}$$

where t is time, $\overline{u}$, $\overline{v}$, and $\overline{w}$ are displacements along the x_1, x_2 and ζ coordinates, respectively, u, v and w are the displacements of a point on the middle surface and ϕ_1 and ϕ_2 are the rotations at $\zeta = 0$ of normal to the mid-surface with respect to the x_2 and x_1 axes, respectively. The functions φ_1, φ_2, θ_1 and θ_2 have system dependent values and will be determined by using the condition that the transverse shear stresses vanish on the plate top and bottom. The displacement field in equation (1) is to be dictated by the desire to represent the transverse shear strains by quadratic functions of the thickness coordinate ζ and by the requirement that the transverse normal strains be zero.

Incorporating the above conditions into equation (1), we obtain [28]

$$\begin{aligned}\overline{u} &= u + \zeta\phi_1 + \zeta^3(4/3h^2)[-\phi_1 - (\partial w/\partial x_1)],\\ \overline{v} &= v + \zeta\phi_2 + \zeta^3(4/3h^2)[-\phi_2 - (\partial w/\partial x_2)],\\ \overline{w} &= w,\end{aligned} \tag{2}$$

where h is the thickness of the laminate.

This displacement field, given by equation (2), is used to compute the strains and stresses at any point within the laminate.

For deterministic finite element analysis with the displacement field mentioned above, it is necessary to employ an element with C^1 continuity. Realizing the computational difficulties associated with C^1 continuity elements, the derivatives of the out-of-plane displacement are themselves considered as separate independent degrees of freedom (DOFs). Thus, 5 DOFs per node with C^1 continuity element is transformed into a C^0 continuity element with 7 DOFs per node.

To make it suitable for C^0 continuity, the displacement field equation (2) needs to be modified. The displacement field may be expressed using functions as coefficient [29]

$$\begin{aligned}\overline{u} &= u + f_1(\zeta)\phi_1 + f_2(\zeta)\theta_1,\\ \overline{v} &= v + f_1(\zeta)\phi_2 + f_2(\zeta)\theta_2,\\ \overline{w} &= w,\end{aligned} \tag{3}$$

where, $\theta_1 = \partial w/\partial x_1$ and $\theta_2 = \partial w/\partial x_2$.

The displacement functions $f_1(\zeta)$ and $f_2(\zeta)$ are represented as $f_1(\zeta) = C_1\zeta - C_2\zeta^3$ and $f_2(\zeta) = -C_4\zeta^3$ with $C_1 = 1$; $C_2 = C_4 = 4/3h^2$.

The displacement vector for the model is,

$$\boldsymbol{\Lambda} = (u\ v\ w\ \theta_2\ \theta_1\ \phi_2\ \phi_1)^T. \tag{4}$$

2.2. *Strain–displacement relations*

The generalized strain–displacement relations are:

$$\begin{aligned}&\varepsilon_1^0 = \partial u/\partial x_1, \quad \varepsilon_2^0 = \partial v/\partial x_2, \quad \varepsilon_4^0 = C_1\phi_2 + \partial w/\partial x_2,\\ &\varepsilon_5^0 = C_1\phi_1 + \partial w/\partial x_1, \quad \varepsilon_6^0 = \partial u/\partial x_2 + \partial v/\partial x_1,\\ &\kappa_1^0 = C_1(\partial\phi_1/\partial x_1), \quad \kappa_2^0 = C_1(\partial\phi_2/\partial x_2), \quad \kappa_6^0 = C_1(\partial\phi_2/\partial x_1 + \partial\phi_1/\partial x_2),\\ &\kappa_4^1 = -3C_2\phi_2 - 3C_4\theta_2, \quad \kappa_5^1 = -3C_2\phi_1 - 3C_4\theta_1,\\ &\kappa_1^2 = -C_2(\partial\phi_1/\partial x_1) - C_4(\partial\theta_1/\partial x_1), \quad \kappa_2^2 = -C_2(\partial\phi_2/\partial x_2) - C_4(\partial\theta_2/\partial x_2),\\ &\kappa_6^2 = -C_2(\partial\phi_2/\partial x_1 + \partial\phi_1/\partial x_2) - C_4(\partial\theta_1/\partial x_2 + \partial\theta_2/\partial x_1).\end{aligned} \tag{5}$$

2.3. Strain energy of the laminate

The elastic strain energy of a laminated composite plate undergoing deformation under the action of external loads is,

$$U = \frac{1}{2} \int_A \bar{\varepsilon}^{-T} \mathbf{D} \bar{\varepsilon} \, \mathrm{d}A, \tag{6}$$

where

$$\bar{\boldsymbol{\varepsilon}} = \left\{ \varepsilon_1^0 \; \varepsilon_2^0 \; \varepsilon_6^0 \; \kappa_1^0 \; \kappa_2^0 \; \kappa_6^0 \; \kappa_1^2 \; \kappa_2^2 \; \kappa_6^2 \; \varepsilon_4^0 \; \varepsilon_5^0 \; \kappa_4^1 \; \kappa_5^1 \right\}^T, \tag{7}$$

$$\mathbf{D} = \begin{bmatrix} \mathbf{A1} & \mathbf{B1} & \mathbf{E} & \mathbf{0} & \mathbf{0} \\ \mathbf{B1} & \mathbf{D1} & \mathbf{F1} & \mathbf{0} & \mathbf{0} \\ \mathbf{E} & \mathbf{F1} & \mathbf{H} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A2} & \mathbf{D2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{D2} & \mathbf{F2} \end{bmatrix}, \tag{8}$$

with

$$(A1_{ij}, B1_{ij}, D1_{ij}, E_{ij}, F1_{ij}, H_{ij}) = \sum_{k=1}^{N} \int_{\zeta_{k-1}}^{\zeta_k} Q_{ij}^{(k)} \left(1, \zeta, \zeta^2, \zeta^3, \zeta^4, \zeta^6\right) \mathrm{d}\zeta,$$
$$i, j = 1, 2, 6, \tag{9}$$

$$(A2_{ij}, D2_{ij}, F2_{ij}) = \sum_{k=1}^{N} \int_{\zeta_{k-1}}^{\zeta_k} Q_{ij}^{(k)} \left(1, \zeta^2, \zeta^4\right) \mathrm{d}\zeta, \quad i, j = 4, 5. \tag{10}$$

2.4. Kinetic energy

The kinetic energy of a vibrating plate in bending within domain of small displacement is

$$T = \frac{1}{2} \int_A \left(\sum_{k=1}^{NL} \int_{\zeta_{k-1}}^{\zeta_k} \rho^{(k)} \dot{\hat{\mathbf{u}}}^T \dot{\hat{\mathbf{u}}} \right) \mathrm{d}\zeta \, \mathrm{d}A, \tag{11}$$

where $\hat{\mathbf{u}}$ is the global displacement vector given by

$$\hat{\mathbf{u}} = \{ \overline{u} \; \overline{v} \; \overline{w} \}. \tag{12}$$

Using equation (3), the above relation can be written as

$$\hat{\mathbf{u}} = \overline{\mathbf{N}} \boldsymbol{\Lambda}, \tag{13}$$

where

$$\overline{\mathbf{N}} = \begin{bmatrix} 1 & 0 & 0 & 0 & f_2(\zeta) & 0 & f_1(\zeta) \\ 0 & 1 & 0 & f_2(\zeta) & 0 & f_1(\zeta) & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{14}$$

Substituting equation (13) in equation (11) gives

$$T = \frac{1}{2}\int_A \left(\sum_{k=1}^{NL}\int_{\zeta_{k-1}}^{\zeta_k} \rho^{(k)}\dot{\mathbf{\Lambda}}^T\,\overline{\mathbf{N}}^T\,\overline{\mathbf{N}}\dot{\mathbf{\Lambda}}\,\mathrm{d}\zeta\right)\mathrm{d}A = \frac{1}{2}\int_A \dot{\mathbf{\Lambda}}^T\mathbf{m}\dot{\mathbf{\Lambda}}\,\mathrm{d}A, \tag{15}$$

where $\mathbf{m}$ is the inertia matrix and is given by

$$\mathbf{m} = \sum_{k=1}^{NL}\int_{\zeta_{k-1}}^{\zeta_k} \rho^{(k)}\overline{\mathbf{N}}^T\,\overline{\mathbf{N}}\,\mathrm{d}\zeta. \tag{16}$$

2.5. Finite Element Model (FEM)

Generally the analyst looks for exact solutions. However, such exact closed form or series type solutions, which employ traditional variational methods, become complex and are very difficult if not impossible to obtain for structures with complex geometry, arbitrary discontinuous loads, or discontinuous material properties or boundary conditions. With the advent of fast computers, the FEM has been found to be a very versatile tool for solving such complex, real life problems. In the present study, a nine noded isoparametric Lagrangian element has been employed.

In the finite element method, the domain is discretized into a set of finite elements. Over each of these elements, the displacement vector $\mathbf{\Lambda}$ is represented as,

$$\mathbf{\Lambda} = \sum_{i=1}^{NN}\varphi_i\mathbf{\Lambda}_i, \tag{17}$$

where φ_i is the interpolation (shape) function for the ith node [30, 31], $\mathbf{\Lambda}_i$ is the vector of unknown displacements for the ith node and NN is the number of nodes per element.

The strain vector given in equations (5) and (7) can be written as

$$\bar{\boldsymbol{\varepsilon}} = \Re\mathbf{\Lambda}, \tag{18}$$

where, $\Re$ is a matrix of differential operators.

2.5.1. Mechanical strain energy. The functional is computed for each element and then summed over all the elements in the domain to get the total functional for the domain. Following this, equation (6) can be written as

$$U = \sum_{e=1}^{NE}U^{(e)} = \sum_{e=1}^{NE}\frac{1}{2}\int_{A^{(e)}}\bar{\varepsilon}^T D\bar{\varepsilon}\,\mathrm{d}A, \tag{19}$$

where, NE is the number of elements.

From equations (18) and (19), we get

$$U^{(e)} = \frac{1}{2}\int_{A^{(e)}}\mathbf{\Lambda}^{\mathbf{T}}\Re^{\mathbf{T}}\mathbf{D}\Re\mathbf{\Lambda}\,\mathrm{d}A. \tag{20}$$

Further, substituting equation (17) in the above equation it can be put in the form

$$U^{(e)} = \frac{1}{2}\int_{A^{(e)}} \mathbf{\Lambda}^{\mathbf{T(e)}}\mathbf{B}^{\mathbf{T}}\mathbf{D}\mathbf{B}\mathbf{\Lambda}^{(e)}\,\mathrm{d}A = \mathbf{\Lambda}^{\mathbf{T(e)}}\mathbf{K}^{\mathbf{(e)}}\mathbf{\Lambda}^{(e)}, \tag{21}$$

in which $\mathbf{K}^{\mathbf{(e)}} = (1/2)\int_{A^{(e)}} \mathbf{B}^{\mathbf{T}}\mathbf{D}\mathbf{B}\,\mathrm{d}A$ with $\mathbf{B} = \Re\varphi$,
and

$$U = \mathbf{q}^{*\mathbf{T}}\mathbf{K}\mathbf{q}^{*} \tag{22}$$

with $\mathbf{K}$ as the global stiffness matrix and $\mathbf{q}^*$ is the global displacement vector.

2.5.2. Kinetic energy. Adopting similar steps as discussed in Subsection 2.5.1, equation (15) can be processed and written as

$$T = \dot{\mathbf{q}}^{*\mathbf{T}}\mathbf{M}\dot{\mathbf{q}}^{*}, \tag{23}$$

where $\mathbf{M}$ is the mass matrix.

2.5.3. Governing equations. Using variational principles, the governing equations for free vibration for the system can be derived as:

$$\mathbf{K}\mathbf{q}^{*} + \mathbf{M}\ddot{\mathbf{q}}^{*} = 0. \tag{24}$$

Assuming that the system vibrates in its principal mode, equation (24) can be expressed as

$$(\mathbf{K} - \lambda\mathbf{M})\mathbf{q}^{*} = 0, \tag{25}$$

where $\lambda = \omega^{*2}$ with ω^* as the natural frequency of vibration.

If $\mathbf{M}$ is positive definite, it is always possible to transform equation (25) into a standard eigenvalue problem.

$$\mathbf{A}\mathbf{q}^{*} = \lambda\mathbf{q}^{*}, \tag{26}$$

where

$$\mathbf{A} = \mathbf{M}^{-1}\mathbf{K}. \tag{27}$$

3. METHOD OF SOLUTION FOR PROBABILISTIC PROBLEM

Equation (26) is the characteristic equation for free vibration of laminated composite plates. The elements of matrix $\mathbf{A}$ are random because the material properties are random. Consequently, the natural frequencies and its mode shapes are also random. Therefore, a straightforward solution, as in a deterministic case would not be possible. The method of solution of this random equation is presented in the next section.

The method of solution presented in Ref. [19] would also be applicable in the present case. However, due to large size of the stiffness matrix arising from the finite element method, a more efficient computational scheme based on left and right eigenvectors is adopted here.

The random variables may be split up without any loss of generality [19] into its mean and zero mean random parts

$$b_i = \overline{b}_i + b_i^r, \quad \mathbf{A} = \overline{\mathbf{A}} + \mathbf{A}^\mathbf{r}, \quad \lambda_j = \overline{\lambda}_j + \lambda_j^r, \quad \mathbf{q}_j^* = \mathbf{q}_j^* + \mathbf{q}_j^*, \quad \omega_j^* = \overline{\omega}_j^* + \omega_j^{*r}, \tag{28}$$

where, b_i's are the material properties, $\mathbf{q}^*$ is the global displacement vector with

$$\lambda_j = \omega_j^2, \text{ giving } \quad \overline{\lambda}_j = \overline{\omega}_j^2 \quad \text{and} \quad \lambda_j = \overline{\omega}_j^2 + \overline{\omega}_j^r + 2\overline{\omega}_j \omega_j^r.$$

Substituting equation (28) into equation (26) and keeping only terms up to the first order, one obtains

$$\text{Zeroth order:} \quad \overline{\mathbf{A}}\,\overline{\mathbf{q}}_j^* = \overline{\lambda}_j \overline{\mathbf{q}}_j^*. \tag{29}$$

$$\text{First order:} \quad \overline{\mathbf{A}}\,\mathbf{q}_j^{*r} + \mathbf{A}^\mathbf{r}\overline{\mathbf{q}}_j^* = \overline{\lambda}_j \mathbf{q}_j^{*r} + \lambda_j^r \mathbf{q}_j^{*r}. \tag{30}$$

The solution of equation (29) can be found out by deterministic procedure [32]. In equation (30), $\mathbf{A}^\mathbf{r}$ can be obtained from equation (27) making use of equation (28). Following the same procedure of separation of the first two equations, the expression for $\mathbf{A}^\mathbf{r}$ is given as

$$\mathbf{A}^\mathbf{r} = \mathbf{M}^{-1}\mathbf{K}^r. \tag{31}$$

The jth first order random vector can be written as [19, 32]

$$\mathbf{q}_j^{*r} = \sum_{l=1}^{N} C_{lj}^r \overline{\mathbf{q}}_j^*, \quad l \neq j, \quad C_{jj}^r = 0. \tag{32}$$

Substituting equation (32) in the first order equation (30), pre-multiplying by $\overline{\mathbf{q}}_j^T$ and making use of the orthogonality conditions [32, 33], one gets

$$\lambda_j^r = \overline{\mathbf{q}}_j^\mathbf{T} \mathbf{A}^\mathbf{r} \overline{\mathbf{q}}_j^*, \tag{33}$$

where the eigenvectors $\overline{\mathbf{q}}_j$ and $\overline{\mathbf{q}}_j^*$ are the left and right eigenvectors of $\overline{\mathbf{A}}$, respectively.

Assuming the random variation to be small in comparison to the mean value [19], the random variables, $\lambda, \mathbf{q}_j^*, \mathbf{q}_j$ and $\mathbf{A}$ can be represented by Taylor's series expansion about $\overline{b}_i$, $i = 1, 2, \ldots, m$. Keeping only first order terms, the relations are:

$$\lambda_j^r = \sum_{i=1}^{m} \overline{\lambda}_{j,i} b_i^r, \quad \mathbf{q}_j^r = \sum_{i=1}^{m} \overline{\mathbf{q}}_{j,i} b_i^r, \quad \mathbf{q}_j^{*r} = \sum_{i=1}^{m} \overline{\mathbf{q}}_{j,i}^* b_i^r, \quad \mathbf{A}^\mathbf{r} = \sum_{i=1}^{m} \overline{\mathbf{A}}_{,i} b_i^r, \tag{34}$$

where i denotes the partial differentiation with respect to b_i evaluated at $\overline{b}_i$.

On substitution of equation (34) into equation (33) and comparison of the coefficients of the basic random variable b_i^r gives

$$\overline{\lambda}_{j,i} = \overline{\mathbf{q}}_j \mathbf{A},_i \overline{\mathbf{q}}_j^*. \tag{35}$$

The variance of the eigenvalue can now be written as [19, 34]

$$Var(\lambda_j) = \sum_{i=1}^{m} \sum_{k=1}^{m} \overline{\lambda}_{j,i} \overline{\lambda}_{j,k} Cov(b_i, b_k), \tag{36}$$

where $Cov(b_i, b_k)$ is the covariance between b_i and b_k.

4. RESULTS AND DISCUSSION

Using the technique presented in the previous section, second order statistics of the fundamental frequency of graphite-epoxy plates have been evaluated with different support conditions. The material properties of the composite lamina have been modeled as the basic random variables. A nine-noded Lagrangian isoparametric element, with 63 DOFs per element for the HSDT model has been used for discretizing the laminated plate. Results have been computed by employing the full (3 × 3) integration rule for thick plates and reduced (2 × 2) integration rule for thin plates. Based on convergence study conducted for the second order statistics of the fundamental frequency, a (5 × 5) mesh has been used throughout the study. The results are presented in terms of non-dimensional frequency, ω defined as:

$$\omega = \left(\omega^* a^2 \sqrt{\rho/\overline{E}_{22}}\right) \Big/ h,$$

in which ω^* is the dimensional fundamental frequency. The overbar denotes the mean value of that quantity.

Various combinations of edge support conditions of clamped (C), free (F) and simple-supports (S) have been used for the investigation. The boundary conditions for the plate are

Clamped edge: $u = v = w = \phi_1 = \phi_2 = \theta_1 = \theta_2 = 0.$

Free edge: $u \neq v \neq w \neq \phi_1 \neq \phi_2 \neq \theta_1 \neq \theta_2 \neq 0.$

Simply supported edge: $v = w = \theta_2 = \phi_2 = 0$ or $u = w = \theta_1 = \phi_1 = 0.$

The following mean material properties have been used in the present investigation:

$$\overline{E}_{11} = 25\overline{E}_{22}, \overline{G}_{12} = \overline{G}_{13} = 0.5\overline{E}_{22}, \overline{G}_{23} = 0.2\overline{E}_{22}, \overline{\nu}_{12} = 0.25, \text{ and } \rho = 1.$$

4.1. Validation

Figure 1 shows the comparison of the probabilistic finite element results with MCS and that obtained from a stochastic classical approach (SCA) [19] for [0/90/90/0] laminated square plate with all edges simply supported for $a/h = 10$ [19]. The

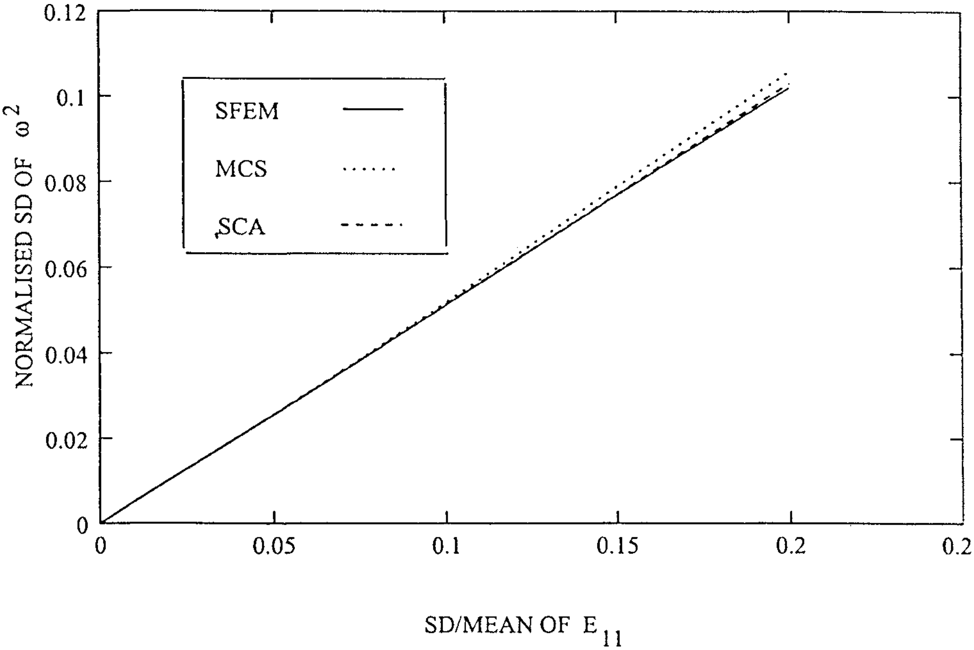

Figure 1. Comparison of the probabilistic finite element results with MCS and SCA for [0/90/90/0] laminated SSSS square plate, with $a/h = 10$.

Table 1.
Comparisons of nondimensionalized mean fundamental frequency for a square plate, $\overline{\omega} = (\overline{\omega}^* a^2 \sqrt{\rho/\overline{E}_{22}})/h$, for all edges simply supported (SSSS)

Methods	Nondimensionalized mean fundamental frequency, $\overline{\omega}$			
	[0/90]		[0/90/90/0]	
	$a/h = 10$	$a/h = 100$	$a/h = 10$	$a/h = 100$
SCA [30]	8.98	10.48	11.77	15.28
SFEM	8.934	11.40	11.68	11.28

number of samples used for the MCS study that gives convergence is 15 000. It is observed that the results are in good agreement with the MCS and SCA. However, the SFEM results show more difference with MCS in comparison to SCA. This may be attributed to the larger numerical round-off errors involved with SFEM. For the range of scatter in the materials considered, the SD of square of the fundamental frequency varies linearly with material properties.

Table 1 shows the comparison of the nondimensionalized mean fundamental frequency obtained using finite element method and classical approach for a square laminate with $a/h = 10$ and 100 for the two stacking sequences [0/90] and [0/90/90/0]. It has been observed that for $a/h = 10$, the full numerical integration scheme gives accurate results. However, for $a/h = 100$, the reduced numerical integration gives accurate results. This may be due to shear locking in thin plates. This conclusion is in line with existing literature. The results are in good agreement.

4.2. Second order statistics

Second order statistics of fundamental frequency have been computed for a graphite–epoxy square plate with $a/h = 5$ and 10 for CCCC, SSSS and CFCF boundary conditions. The stacking sequence considered for the study is [0/45/ −45/90].

4.2.1. Frequency expectation. Table 2 shows the non-dimensional mean fundamental frequency ($\overline{\varpi}$), for the laminated composite square plates for $a/h = 5$ and 10 with CCCC, CFCF and SSSS boundary conditions. As expected, the CCCC panels give higher values as compared to other support conditions. The fundamental frequency increases as a/h ratio increases.

4.2.2. Frequency variance. Figures 2a and 2b show the SD of square of the fundamental frequency of the square plate with SD of material properties varying simultaneously for $a/h = 5$ and 10. From the examination of results, it is observed that for $a/h = 5$, CCCC panel shows smaller scatter in frequency as compared to SSSS and CFCF panels. The difference between CCCC and CFCF are small, while the difference is large between CCCC and SSSS panels. For $a/h = 10$, the CCCC and CFCF panels show almost the same sensitivity. Other observations are similar to the panel of $a/h = 5$. The sensitivity of panels changes as the a/h ratio changes. These changes are appreciable for CCCC and CFCF panels as compared to SSSS panel.

Figures 3a–3f present the variation of normalized SD of square of the fundamental frequency (ω^2) for the square plate with $a/h = 5$ for SD of material property changing one at a time, other values kept constant, while Figs 4a–4f are for $a/h = 10$ with different edge conditions. For a panel with $a/h = 5$, it may be noted that the response of CCCC panel is least affected by scatter in E_{11}, E_{22}, G_{12} and ν_{12}, while it is most affected by scatter in the out-of-plane shear modulii, G_{23}

Table 2.
Nondimensionalized mean fundamental frequency, $\overline{\varpi} = (\overline{\varpi}^* a^2 \sqrt{\rho/\overline{E}_{22}})/h$ for a [0/45/−45/90] square plate with different boundary conditions

Support conditions	Nondimensionalized mean fundamental frequency, $\overline{\varpi}$	
	$a/h = 5$	$a/h = 10$
CCCC	10.74	16.40
CFCF	7.38	11.10
SSSS	7.17	8.89

and G_{13}. Further, it can also be noted that the CCCC and CFCF panels show closeness in scatter in frequency against variation in material properties. However, this is not so with the scatter in Poisson's ratio, ν_{12}. The CCCC and CFCF panels are most sensitive to G_{13}, while SSSS panel is most sensitive to E_{11}. The panels are least affected by changes in the Poisson's ratio.

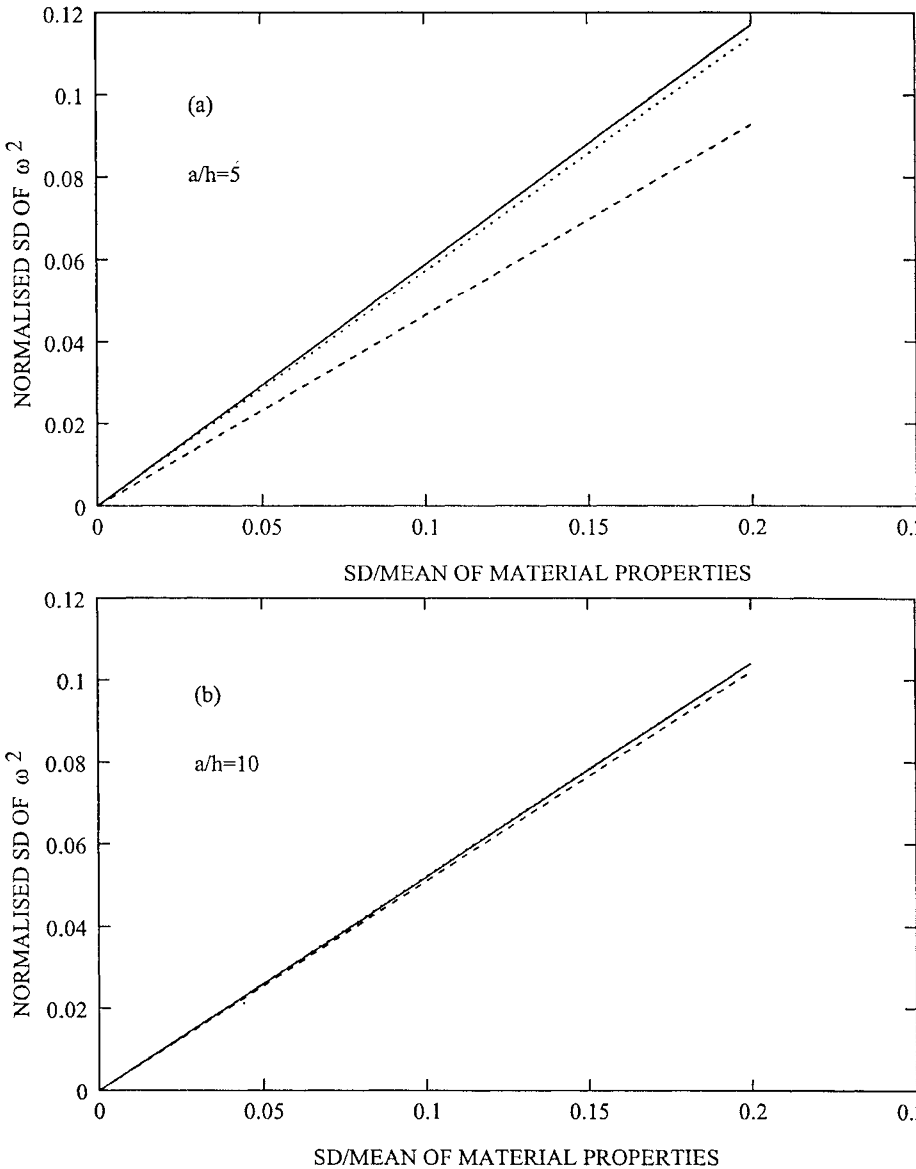

Figure 2. Variation of square of the fundamental frequency of a [0/45/−45/90] laminated composite square plate with SD of material properties varying simultaneously. Key: (—) SSSS, (· · ·) CFCF, (- - -) CCCC.

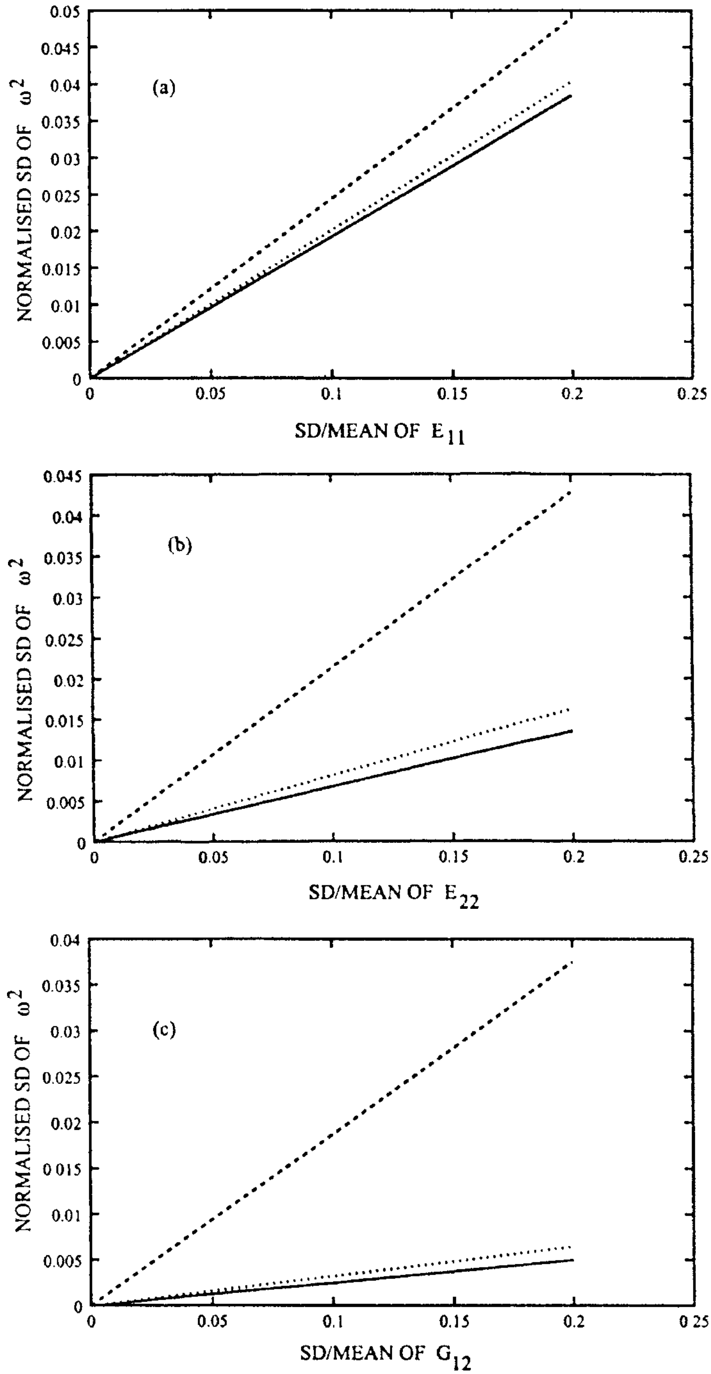

Figure 3. Variation of normalized SD of square of the fundamental frequency of a [0/45/−45/90] laminated square plate with SD of material properties changing at a time for $a/h = 5$. Key: as in Fig. 2.

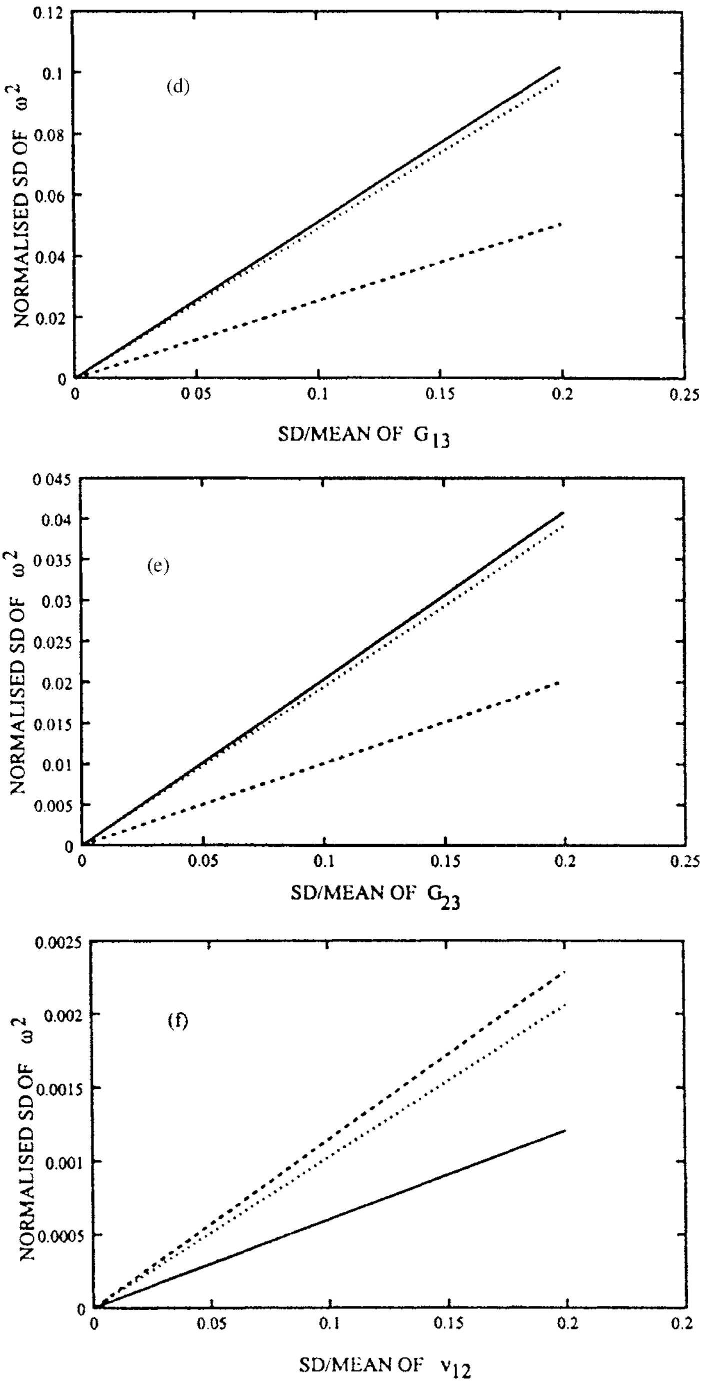

Figure 3. (Continued).

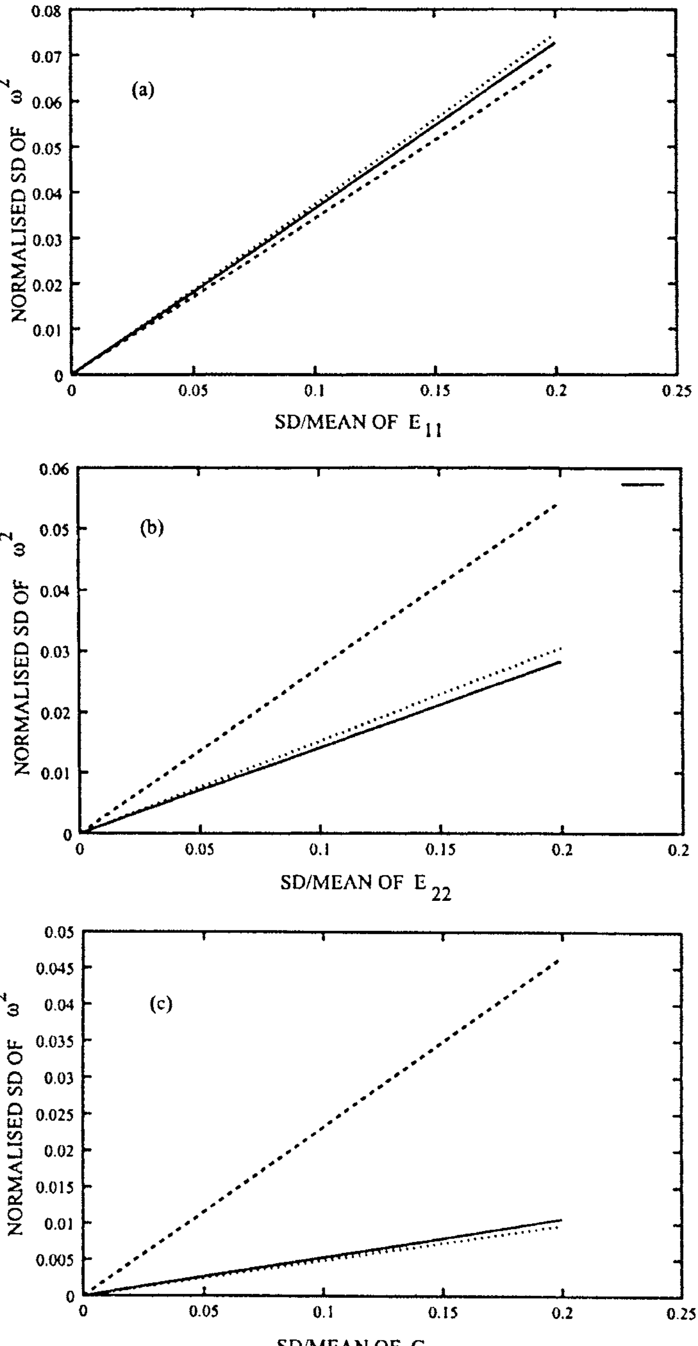

Figure 4. Variation of normalized SD of square of the fundamental frequency of a [0/45/−45/90] laminated square plate with SD of material properties changing at a time for $a/h = 10$. Key: as in Fig. 2.

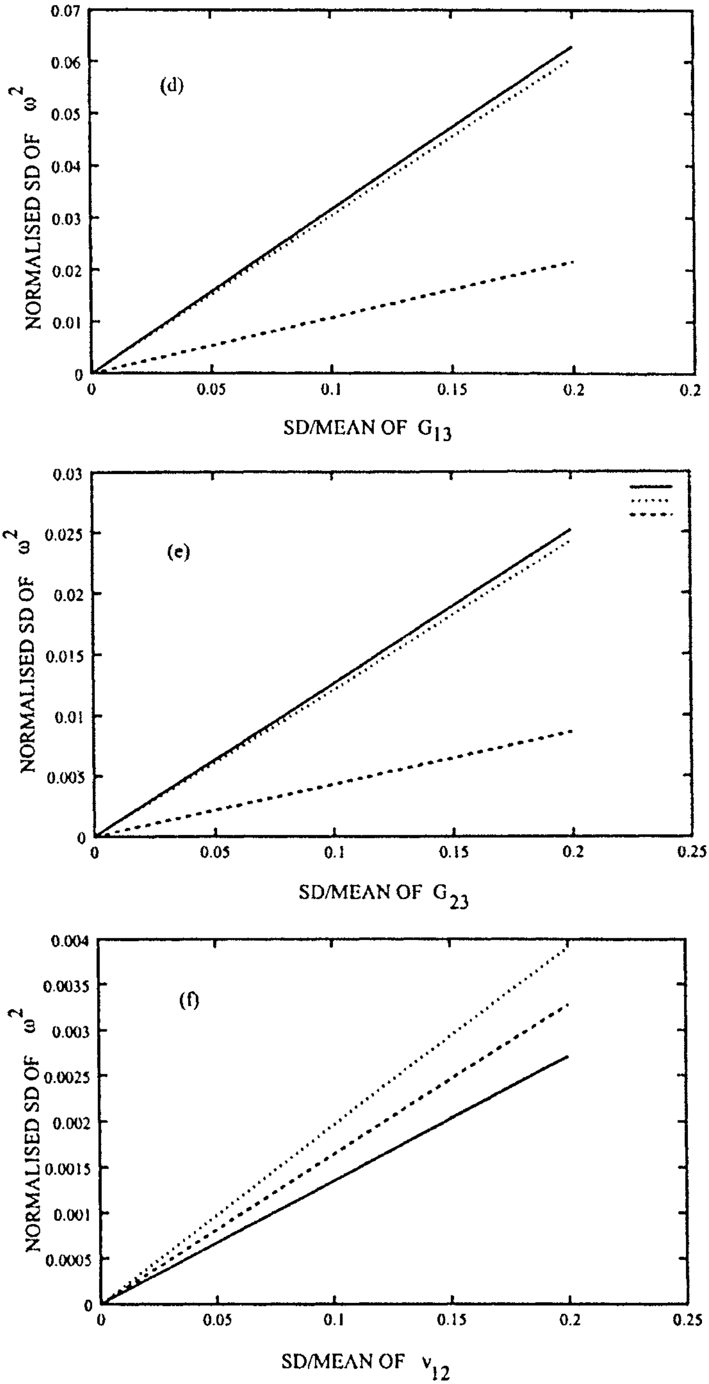

Figure 4. (Continued).

For $a/h = 10$, the panels are most sensitive to SD of the E_{11}, while they are least affected by the SD of Poisson's ratio. The effects of G_{13} on scatter of the fundamental frequency of CCCC and CFCF panels are considerable. The CFCF panel is most sensitive as compared to other edge conditions against individual variation of material properties. The results reveal that the sensitivity of CCCC and CFCF panels are of the same order for all material property variations except in the case of Poisson's ratio. The results also reveal that SSSS panels do not show that much sensitivity against variation of the out-of-plane shear modulii as compared to the other end conditions for the panel.

5. CONCLUSIONS

In the present work, the deterministic finite element method has been used in conjunction with first order perturbation technique to analyse the free vibration of composite thick as well moderately thick plates with different boundary conditions having random material properties. Higher order shear deformation theory has been used to model the plates. A computationally efficient analysis procedure has been presented. The second order statistics of the fundamental frequency of a laminated composite plate have been obtained. The main conclusions are:

- The scatter in the fundamental frequency increases with the individual variations of E_{11}, E_{22}, G_{12} and ν_{12} as the side to thickness ratio increases, while the scatter decreases with G_{13} and G_{23}.
- CCCC panel with $a/h = 5$ is most sensitive for changes in fundamental frequency with simultaneous changes in the material properties, while SSSS panel with $a/h = 5$ is least sensitive.
- The dominance of material property on the scatter in the fundamental frequency changes from E_{11} to G_{13} as a/h ratio changes from 10 to 5.
- The composite panels are least affected with changes in the Poisson' s ratio.

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